Unit – 1
Linear system of equations

Rank of the matrix:

Let A be m × n matrix, matrix A is said to be of rank r when

(i) It has at least one non-zero minor of order r,

(ii) Every minor of order higher than r vanishes.

(or)

The rank of a matrix is the largest order of any non-vanishing minor of the matrix. The rank of a matrix A shall be denoted by $\rho(A)$.

Note: 1. If a matrix has a non-zero minor of order r, its rank is > r.

2. If all minors of a matrix of order r + 1 are zero, its rank is < r.

Ex1: find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

det A = $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix} = 0$

rank of A $\neq 3$.

Consider a minor of order 2 = $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0$

Rank of the matrix A = 2.

Ex2: find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

det A = $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{vmatrix} = 24 \neq 0$

Rank of the matrix A = 3.

Echelon form of a matrix:

A matrix is said to be in echelon form if it has the following properties

(1) Zero rows, if any, are below any non-zero row.
(2) The first non-zero entry in each non-zero row is equal to 1.
(3) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Conditional (2) is optional.

Ex1:	Reduce the matrix A = $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into echelon form and hence find its rank
Sol:	Consider A = $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$ $R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & -3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$ $R_4 \leftrightarrow R_4 - R_2$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & -3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$ $R_4 \leftrightarrow R_4 - R_3$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & -3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ This is in Echelon form and the number of non-zero rows is 3. Rank of A = ρ(A) = 3
Ex2:	Reduce the matrix A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ into echelon form and hence find its rank
Sol:	Consider A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

	$R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$ $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ $R_4 \to R_4 - R_2, R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ This is in Echelon form and the number of non-zero rows is 2. Rank of A = ρ(A) = 2

The following problems are discussed in the class work:

1	Reduce the matrix A = $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ into echelon form and hence find its rank
2	Reduce the matrix A = $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ into echelon form and hence find its rank
3	Find the rank of the matrix A= $\begin{bmatrix} 3 & 4 & 5 & 7 \\ 5 & 4 & 6 & 8 \\ 5 & 6 & 7 & 9 \\ 15 & 16 & 17 & 19 \end{bmatrix}$ .by reducing to Echelon form

Normal form:

Every m × n matrix of rank r can be reduced to the form I_r, $\begin{bmatrix} I_r & 0 \end{bmatrix}$ or $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite chain of elementary row or column operations, where I_r is the r-rowed unit matrix.

The above form is called "normal form" or "canonical form "of a matrix.

Ex1:	Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$, by reducing it to be the normal form.
Sol:	Let A = $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$ $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$ $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

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$R_4 \to R_4 - R_2 \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$R_4 \to R_4 - R_3 \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \to C_2 - 2C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - 2C_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \to C_3 - C_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of the matrix is 3 and This is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

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<tr><td>Ex2:</td><td>

Reduce the following matrix to Normal form and hence find it's rank $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

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<tr><td>Sol:</td><td>

Sol: let A = $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$R_2 \to R_2 - 4R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - R_1 \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$

$C_2 \to C_2 + C_3 \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$C_2 \to C_2 + C_1,\ C_4 \to C_4 - 3C_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$

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	$R_4 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $C_3 \to C_3 + C_2,\ C_4 \to C_4 - C_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & -16 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to 6R_3 + R_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & -16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $C_2 \to \frac{C_2}{6}, C_3 \to \frac{C_3}{6},\ C_4 \to \frac{C_4}{-16} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $C_3 \to C_3 - C_2,\ C_4 \to C_4 - C_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ This is of the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ and the rank of the matrix = 2

The following problems are discussed in the class work:

1	Reduce the matrix A= $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ into Normal Form and hence find its Rank.
2	Reduce the matrix A = $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ into normal form and hence find its rank
3	Find the rank of the matrix A= $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ by reducing to normal form

Elementary matrix:

It is a matrix obtained from a unit matrix by a single elementary transformation.

Ex: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a elementary matrix obtained from unit matrix by applying elementary transformation $R_1 \leftrightarrow R_2$.

Normal form with PAQ:

Every elementary row (column) transformation of a matrix can be obtained by pre-multiplication (post-multiplication) with corresponding elementary matrix.

Ex1:	Obtain non-singular matrices P and Q such that PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where A = $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ and hence obtain its rank.
Sol:	Given, A = $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ This can be written as A = $I_3 A I_3$ $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \rightarrow R_3 + R_2$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - 2C_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $C_3 \rightarrow C_3 - C_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ This is of the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ = PAQ, where P = $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and Q= $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ Hence rank of the matrix = 2
Ex2:	Find the non-singular matrices P and Q such that PAQ is in normal form A=

	$\begin{bmatrix}1 & 3 & 6 & -1\\ 1 & 4 & 5 & 1\\ 1 & 5 & 4 & 3\end{bmatrix}$
Sol:	Given, A = $\begin{bmatrix}1 & 3 & 6 & -1\\ 1 & 4 & 5 & 1\\ 1 & 5 & 4 & 3\end{bmatrix}$ This can be written as A = $I_3\ A\ I_3$ $\begin{bmatrix}1 & 3 & 6 & -1\\ 1 & 4 & 5 & 1\\ 1 & 5 & 4 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ $\begin{bmatrix}1 & 3 & 6 & -1\\ 0 & 1 & -1 & 2\\ 0 & 2 & -2 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} A$ $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ $R_3 \to R_3 - 2R_2$ $\begin{bmatrix}1 & 3 & 6 & -1\\ 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ $C_2 \to C_2 - 3C_1, C_3 \to C_3 - 6C_{1,} C_4 \to C_4 + C_1$ $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix} A \begin{bmatrix}1 & -3 & -6 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ $C_3 \to C_3 + C_2, C_4 \to C_4 - 2C_2$ $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix} A \begin{bmatrix}1 & -3 & -9 & 7\\ 0 & 1 & 1 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ This is of the form $\begin{bmatrix}I_2 & 0\\ 0 & 0\end{bmatrix}$ = PAQ, where P = $\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}$ and Q= $\begin{bmatrix}1 & -3 & -9 & 7\\ 0 & 1 & 1 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ Hence rank of the matrix = 2
3	Find the non-singular matrices P and Q such that PAQ is in normal form

	A= $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$
Sol:	we write A = $I_3\, A\, I_4$ $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ By applying elementary transformations, We get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1/120 & 1/60 \\ -1/24 & 0 & 1/12 \end{bmatrix} A \begin{bmatrix} 1 & -20 & 20 & 0 \\ 0 & 25 & -1 & 0 \\ 0 & -5 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Rank of the matrix = 3

The following problems are discussed in the class work:

1	Find the non-singular matrices P and Q such that PAQ is in normal form A= $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$
2	Obtain non-singular matrices P and Q such that PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where A = $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and hence obtain its rank

The inverse of a matrix by elementary transformations: (Gauss – Jordan method)

Suppose A is a non – singular square matrix of order n.

We write A = I_n A

We apply elementary row operations only to the matrix A and the prefactor I_n of the R.H.S. we get the equation of the form I_n = B A

Here B is the inverse of A.

Ex1:	Given $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find the inverse

Sol:	Let A = $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ We write A = I_3 A $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ $R_1 \rightarrow R_2 + R_1$ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ $R_1 \rightarrow R_3 + R_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ $R_2 \rightarrow \frac{R_2}{2}, R_3 \rightarrow \frac{R_3}{3}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} A$ This is of the form I_3 = B A The inverse of the matrix A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

The following problems are discussed in the class work:

1	Find the inverse of matrix A= $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ using elementary transformation.
2	Find the inverse of A = $\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ using elementary transformation.

System of Linear simultaneous equations:

Def: An equations of the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots. + a_nx_n = b$ where $x_1, x_2, \ldots x_n$ are unknowns and $a_1, a_2, \ldots, a_n$, b are constants is called a linear equations in n unknowns.

Def: consider the system of m linear equations in n unknowns $x_1, x_2, \ldots x_n$are

$$
\begin{aligned}
a_{11}x_1 &\quad + a_{12}x_2 &\quad + a_{13}x_3 + \ldots + a_{1n}x_n &\quad = &\quad b_1 \\
a_{21}x_1 &\quad + a_{22}x_2 &\quad + a_{23}x_3 + \ldots + a_{2n}x_n &\quad = &\quad b_2 \\
a_{31}x_1 &\quad + a_{32}x_2 &\quad + a_{33}x_3 + \ldots + a_{3n}x_n &\quad = &\quad b_3
\end{aligned}
$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{mn}x_n = b_m$$

Where a a_{ij}'s and $b_1, b_{2,}\ldots$ b_n are constants. The above system is called the system of simultaneous equations.

The above system can be in written in the form of matrix A X = B, where A = $\lfloor a_{ij} \rfloor$, X = $(x_1, x_2, \ldots, x_n)^T$ and B = $(b_1, b_2, \ldots, b_m)^T$

If B = 0, the system is called a **homogenous** system of linear equations.

The system AX = O is always consistent since X = 0 is always a solution of AX = O.

This solution is called the trivial solution of the system.

If $B \neq 0$, the system is called a **non-homogeneous** system of linear equations.

For $|A| \neq 0$, A^{-1} exist.

The system A X = B is consistent if it has a solution otherwise is said to be inconsistent.

Non – homogenous system:

The system AX = B is consistent i.e, it has a solution if and only if $\rho(A) = \rho(A/B)$

(i) If $\rho(A) = \rho(A/B) = r = n$ then the system has unique solution.

(ii) If $\rho(A) = \rho(A/B) = r < n$ then the system is consistent, but there exists infinite number of solutions. Giving arbitrary values to n – r of the unknowns.

(iii) If $\rho(A) \neq \rho(A/B)$ then the system is inconsistent i.e, the system has no solution.

Ex1:	Solve the system of equations x + y + z = 9, 2x + 5y + 7z = 52, 2x + y – z = 0
Sol:	The given system of equations can be written in matrix form AX = B $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$ The augmented matrix is [A / B] = $\begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

	$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1 \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{bmatrix}$ $R_3 \to 3R_3 + R_2 \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{bmatrix}$ Here $\rho(A) = 3, \rho(A/B) = 3$ We have $\rho(A) = \rho(A/B) = 3 =$ n The system is consistent and it has a unique solution. The equations are x + y + z = 9 3y + 5z = 34 -4z = -20 => z = 5 We have 3y + 25 = 34 => y = 3 And x + 3 + 5 = 9 => x = 1 Solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$
Ex 2:	Show that the system of equations x + y + z = 4, 2x + 5y – 2z = 3, x + 7y -7 z = 5 are not consistent.
Sol:	The given system of equations can be written in matrix form AX = B $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ The augmented matrix is [A / B] = $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$ $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1 \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$ $R_3 \to R_3 - 2R_2 \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$ Here $\rho(A) = 2, \rho(A/B) = 3$ We have $\rho(A) \neq \rho(A/B)$ The given system is not consistent.
Ex 3:	Find for what values of λ the equations x + y + z = 1, x + 2y + 4z = λ, x + 4y + 10 z = λ^2 have a solution and solve then completely in each case.

Sol: The given system of equations can be written in matrix form AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

The augmented matrix is [A / B] = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{bmatrix}$

The given equations will be consistent iff $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

Case (i): if $\lambda = 1$then

[A / B] = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Here $\rho(A) = 2, \rho(A/B) = 2$

The system is consistent and the number of arbitrary constants are n-r = 3-2 =1

the equations are x+ y + z = 1 , y + 3z = 0

Let z = k then y = -3k and x = 2k + 1

X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k+1 \\ -3k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case (ii): if $\lambda = 2$then

[A / B] = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Here $\rho(A) = 2, \rho(A/B) = 2$

The system is consistent and the number of arbitrary constants are n-r = 3-2 =1

	the equations are x+ y + z = 1 , y + 3z = 1 Let z = k then y = 1-3k and x = 2k $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ 1-3k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
Ex 4:	Find the values of a and b for which the equations x + y + z = 3, x + 2y + 2z =6, x + ay +3z = b have (i) No solution (ii) a unique solution (iii) infinite number of solutions.
Sol:	The given system of equations can be written in matrix form AX = B $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$ The augmented matrix is [A / B] = $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-3 \end{bmatrix}$ $R_3 \rightarrow R_3 - 2R_2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{bmatrix}$ **No solution:** ρ(A) ≠ ρ(A/B) Then ρ(A) = 2, ρ(A/B) = 3 So a = 3 and b $\neq 9$ **Unique solution:** ρ(A) = ρ(A/B) = n(unknowns) Then ρ(A) = 3, ρ(A/B) = 3 So a ≠ 3 and b is any value **Infinite number of solutions:** ρ(A) = ρ(A/B) < n(unknowns) Then ρ(A) = 2, ρ(A/B) = 2 and n = 3 So a = 3 and b = 9

The following problems are discussed in the class work:

1	Find whether the following system of equations are consistent. If so solve them x + 2y + 2z = 2, 3x – 2y –z = 5, 2x – 5y + 3z = -4 , x + 4y + 6z =0
2	Find the values of a and b for which the equations x + y + z = 3, x + 2y + 2z =6, x + 9y +az = b have (i) No solution (ii) a unique solution (iii) infinite number of solutions.
3	Solve the system of equations x + y + z = 6, x – y + 2z = 5, 3x + y + z = 8

Homogenous system:

The system AX = O is consistent

(i) If $\rho(A) = r = n$ then the system of equations have only trivial solution

(ii) If $\rho(A) = r < n$ then the system of equations have an infinite number of solutions.

Giving arbitrary values to n – r of the unknowns.

Ex1:	Solve the system of equations x + y +w = 0, y + z = 0, x + y +z + w = 0, x + y + 2z = 0
Sol:	The given system of equations can be written in matrix form AX = O $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ The coefficient matrix is A = $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$ $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$ $R_4 \to R_4 - 2R_3$ $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

	Rank of A = 4 and number of variables = 4 Therefore, the system of zero solution The solutions are x = y = z = w = 0.
Ex2:	Solve completely the system of equations x + 3 y – 2z = 0, 2x – y + 4z = 0, x – 11y + 14z = 0
Sol:	The given system of equations can be written in matrix form AX = O $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ The coefficient matrix is A = $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$ $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1 \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$ $R_3 \rightarrow \frac{R_3}{2} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -7 & 8 \end{bmatrix}$ $R_3 \rightarrow R_3 - R_2 \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ This is an Echelon form. Rank of A = 2 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 2 = 1 The equations are x + 3 y – 2z = 0, -7y + 8z = 0 Let z = k, then y = $\frac{8k}{7}$, x = $\frac{-10k}{7}$ The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (-10k)/7 \\ 8k/7 \\ k \end{bmatrix} = k\begin{bmatrix} (-10)/7 \\ 8/7 \\ 1 \end{bmatrix}$
Ex3:	Solve the system λx + y + z = 0, x + λy + z = 0, x + y + λz = 0 if the system has non –zero solution only.
Sol:	The given system of equations can be written in matrix form AX = O $\begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

	The coefficient matrix is A = $\begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$ if the system has non –zero solution then det A = 0 then we have $\begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$ λ = 1, 1 and -2 Case(i) : put λ = 1 Then $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Rank of A = 1 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 1 = 2 The equations are x + y + z = 0 Let y = c, z = k, then x = $-c - k$ The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -c-k \\ c \\ k \end{bmatrix} = c \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ Case (ii): put λ = -2 then we have $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1 \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$

	$R_3 \to R_3 + R_2 \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ This is an Echelon form. Rank of A = 2 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 2 = 1 The equations are x – 2y + z = 0, - 3y + 3z = 0 => y = z Let z = k, then y = k , x = k The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The following problems are discussed in the class work:

1	Solve completely the system of equations x + 2y – 2z + 3w = 0, x – 2y + z – w = 0, 4x + y – 5z + 8w = 0, 5x – 7y + 2z – w = 0
2	Determine the values of λ for which the following set of equations may possess non – trivial solution: 3x + y – λz = 0, 4x – 2y – 3z = 0, 2λx + 4y + λz = 0

Solutions of linear systems – Direct methods:

1. Gauss Elimination method:

Ex1:	Solve the system of equations 3x + y + 2z = 3, 2x - 3y - z = -3, x + 2y +z = 4
Sol:	The given system of equations can be written in matrix form AX = B $\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$ The augmented matrix is [A / B] = $\begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$ $R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}$

	$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1 \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix}$ $R_3 \rightarrow 7R_3 - 5R_2 \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix}$ The equations are x + y + z = 4 -7y - 3z = -11 -8z = -8 => z = -1 We have -7y + 3 = -11=> y = 2 And x + 2 - 1 = 4 => x = 1 Solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

2. Gauss – Jordan method:

Ex1:	Solve the system of equations 2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16
Sol:	The given system of equations can be written in matrix form AX = B $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$ The augmented matrix is [A / B] = $\begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$ $R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{bmatrix}$ $R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 2R_1 \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{bmatrix}$ $R_1 \rightarrow 5R_1 + R_2, R_3 \rightarrow 10R_3 - 7R_1 \begin{bmatrix} 1 & 0 & -19 & -10 \\ 0 & -10 & -24 & -30 \\ 0 & 0 & -2 & -10 \end{bmatrix}$

	$R_1 \to 2R_1 - 19R_3, R_2 \to R_2 - 12R_3 \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & -10 & 0 & 90 \\ 0 & 0 & -2 & -10 \end{bmatrix}$ $R_2 \to \frac{R_2}{-10}, R_3 \to \frac{R_3}{-2} \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ Which gives x = 7, y = -9, z = 5

Def: Consider the system of equations

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 &+ a_{13}x_3 &= b_1 \\ a_{21}x_1 &+ a_{22}x_2 &+ a_{23}x_3 &= b_2 \\ a_{31}x_1 &+ a_{32}x_2 &+ a_{33}x_3 &= b_3 \end{aligned}$$

Where the diagonal coefficients are not zero and are large compared to other coefficients, such system is called a diagonally dominant system.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others. We now explain three such methods

3. Gauss Jacobi iteration method:

Ex1:	Use gauss - jacobi iteration method to solve the equations 10x + y - z = 11.19, x + l0y + z = 28.08, - x + y + l0z = 35.61, correct to two decimal places.
Sol:	The given system is diagonally dominant, so we can write x = $\frac{1}{10}(11.19 - y + z)$ y = $\frac{1}{10}(28.08 - x - z)$ z = $\frac{1}{10}(35.61 + x - y)$ take the intial values x = 0, y = 0, z = 0 then we get

Variable	Ist approx.	IInd approx.	IIIrd approx.	IVth approx.	Vth approx.
x	1.119	1.19	1.22	1.23	1.23
y	2.808	2.24	2.35	2.34	2.34
z	3.561	3.39	3.45	3.45	3.45

The solution of the given system of equations is x = 1.23, y = 2.34, z = 3.45.

4. Gauss Seidel Iteration method:

Ex1:	Use gauss - seidel iteration method to solve the equations 27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110
Sol:	The given system is diagonally dominant, so we can write

x = $\frac{1}{27}\,(85 - 6y + z)$

y = $\frac{1}{15}\,(72 - 6x - 2z)$

z = $\frac{1}{54}\,(110 - x - y)$

take the intial values y = 0, z = 0 then we get

Variable	Ist approx.	IInd approx.	IIIrd approx.	IVth approx.	Vth approx.
x	3.15	2.43	2.426	2.425	2.425
y	3.54	3.57	3.572	3.573	3.573
z	1.91	1.926	1.926	1.926	1.926

The solution of the given system of equations is x = 2.425, y = 3.573, z = 1.926.

Ex2: Use gauss - seidel iteration method to solve the equations 20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25

Sol: The given system is diagonally dominant, so we can write

x = $\frac{1}{20}\,(17 - y + 2z)$

y = $\frac{1}{20}\,(-18 - 3x + z)$

z = $\frac{1}{20}\,(25 - 2x + 3y)$

take the intial values y = 0, z = 0 then we get

Variable	Ist approx.	IInd approx.	IIIrd approx.	IVth approx.
x	0.8500	1.0025	1	1
y	-1.0275	-0.9998	-1	-1
z	-1.0109	0.9998	1	1

The solution of the given system of equations is x = 1, y = - 1, z = 1.

Ex 3: Apply Gauss-Seidel iterative method to solve the following system of equations $x + 10y + z = 6,\ 10x + y + z = 6,\ x + y + 10z = 6$

Sol: the given system is diagonally dominated

So we can write as x = $\frac{6-y-z}{10}$

y = $\frac{6-z-x}{10}$

z = $\frac{6-y-x}{10}$

take the initial values are y = 0, z = 0

	I approx.	II approx.	III approx.	IV approx.
x	0.6	0.4974	0.4999	0.5
y	0.54	0.5017	0.5	0.5
z	0.486	0.4992	0.5	0.5

The solution is x = 0.5 , y = 0.5, z = 0.5

The following problems are discussed in the class work:

1	Use Gauss elimination method, solve completely the system of equations

	2x + y +2z + w = 6, 6x – 6y + 6z + 12w = 36, 4x + 3y + 3z - 3w = -1, 2x + 2y - z + w = 10
2	Use Gauss Jordan method, solve the system of equations 10x + y + z = 12, x + 10y – z = 10 and x – 2y + 10z = 9
3	Use Gauss Jacobi method, solve the system of equations 10x + y + z = 12, 2x + 10y + z = 13 and 2x + 2y + 10z = 14
4	Use gauss - seidel iteration method to solve the equations 8x -3y + 2z = 20, 4x + 11y - z = 33, 6x + 3y + 12z = 35

Application: Finding the current in an Electrical circuit:

Consider circuits made up of

(i) three passive elements—resistance, inductance, capacitance and

(ii) an active element—voltage source which may be a battery or a generator.

Ohm's law: the current through a conductor between two points is directly proportional to the potential difference across the two points.

Kirchhoff's laws: The formulation of differential equations for an electrical circuit depends on the following two Kirchhoff's laws which are of cardinal importance :

I. The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.

II. The algebraic sum of the currents flowing into (or from) any node is zero.

Electrical circuit: a simple electric circuit is a closed connection of Batteries, Resisters, and wires. An electrical circuit consists of voltage loops and current nodes.

UNIT – 2
EIGEN VALUES AND VECTORS AND QUADRATIC FORMS

If A is a $n \times n$ matrix, then $X \neq \vec{0}$ is said to be an eigenvector of A if there exists a scalar λ such that A $X = \lambda X$

Here The scalar λ is called the eigenvalue or characterstic value or proper value of A and X is called the eigenvector or characerstic vector or proper vector corresponding to the eigenvalue λ.

How do I find eigenvalues of a square matrix?

If A is a $n \times n$ matrix. Let X be an Eigen vector of A corresponding to the eigen value λ

$$A X = \lambda X$$

$$A X - \lambda X = 0$$

$$A X - \lambda I X = 0$$

$$(A - \lambda I) X = 0$$

This is a homogenous system of n equations in n unknowns.

The system has a non-zero solution X, if and only if $\det(A - \lambda I) = 0$

Here det(A – λI) = 0 is also called the characteristic equation of A. this will be a polynomial equation in λ of degree n.

Ex1:	Find the eigen values and eigen vectors of the following matrix $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$
Sol:	The characteristic equation of A is $\lvert A - \lambda I \rvert = 0$ $\Rightarrow \begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{vmatrix} = 0$ $\Rightarrow \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$ $\Rightarrow \lambda = 3, 6, 9$ The eigen values of A are $3, 6, 9$ **Case(i):** put $\lambda = 3$ We have $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$R_2 \to R_2 + R_1 \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is an Echelon form.
Rank of A = 2 and number of variables = 3
Therefore, the system of infinite number of non-zero solutions
The number of arbitrary constants are n – r = 3 – 2 = 1
The equations are 2x - 2y = 0, y + 2z = 0
Let z = k, then y = $-2k$, x = 2k

$$\text{The solution } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

The $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value λ = 3.

Case(ii): put $\lambda = 6$

$$\text{We have } \begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 \begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to 2R_3 - R_2 \begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is an Echelon form.
Rank of A = 2 and number of variables = 3
Therefore, the system of infinite number of non-zero solutions
The number of arbitrary constants are n – r = 3 – 2 = 1
The equations are -x - 2y = 0, 4y + 2z = 0
Let y = k, then z = $-2k$, x = -2k

	The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k \\ k \\ -2k \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ The X = $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to the eigen value λ = 6. **Case(iii):** put $\lambda = 9$ We have $\begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_2 \rightarrow 2R_2 - R_1$ $\begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_3 \rightarrow 2R_3 + R_2$ $\begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ This is an Echelon form. Rank of A = 2 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 2 = 1 The equations are - 4x - 2y = 0, -4y + 4z = 0 Let y = k, then z = k , x = k/2 The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k/2 \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ The X = $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value λ = 9.
Ex2:	Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$
Sol:	The characterstic equation of A is $\lvert A - \lambda I \rvert = 0$ ⇨ $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3 \end{vmatrix} = 0$ ⇨ $(\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$

	$\Rightarrow \lambda = 2, 2, 8$ The eigen values of A are 2, 2, 8. **Case(i):** put $\lambda = 2$ We have $\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_2 \rightarrow 2R_2 + R_1, R_3 \rightarrow 2R_3 - R_1,$ $\begin{bmatrix} -4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ This is an Echelon form. Rank of A = 1 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 1 = 2 The equation is - 4x - 2y + 2z = 0, Let y = k, z = c then x = $\frac{k}{2} - \frac{c}{2}$ The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{k}{2} - \frac{c}{2} \\ k \\ c \end{bmatrix} = k \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ The eigen vector o f A corresponding to the eigen value $\lambda = 2$ is $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ **Case(ii):** put $\lambda = 8$ We have $\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 + R_1,$ $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_3 \rightarrow R_3 - R_2$ $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ This is an Echelon form. Rank of A = 2 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 2 = 1

	The equations are – 3y – 3z = 0, - 2x - 2y + 2z = 0 Let z = k, then y = $-k$, x = 2k The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ The eigen vector o f A corresponding to the eigen value $\lambda = 8$ is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

The following problems are discussed in the class work:

1	Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$
2	Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Properties of Eigen values:

Theorem1: the sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof: the characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22}-\lambda & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn}-\lambda \end{vmatrix} = 0$$

By expanding this, we get

$(a_{11} - \lambda)(a_{22} - \lambda) \ldots (a_{nn} - \lambda) - a_{12}$ (a polynomial of degree n – 2) + a_{13} (a polynomial of degree n – 2) + . . . = 0.

i.e, $(-1)^n (\lambda - a_{11})(\lambda - a_{22}) \ldots (\lambda - a_{nn})$ + a polynomial of degree (n – 2) = 0

i.e, $(-1)^n [\lambda^n - (a_{11} + a_{22} + \ldots + a_{nn}) \lambda^{n-1}$ + a polynomial of degree (n-2)] + a polynomial of degree (n-2) in $\lambda = 0$.

i.e, $(-1)^n \lambda^n + (-1)^{n+1}$ (Trace A) λ^{n-1} + a polynomial of degree (n-2) in $\lambda = 0$

if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the roots of this equation,

Sum of the roots = $-\frac{(-1)^{n+1} Tr(A)}{(-1)^n}$ = Tr(A)

We have $|A - \lambda I| = (-1)^n \lambda^n + \dots + a_0$

Put $\lambda = 0$. Then $|A| = a_0$

$(-1)^n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \dots + a_0 = 0$

=> Product of the roots = $\frac{(-1)^n a_0}{(-1)^n} = a_0 = |A| = \det A$.

Theorem2: if λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value of A^n corresponding to the eigen vector X.

Proof: Since λ is an eigen value of A corresponding to the eigen vector X,

Then $AX = \lambda X$

premultiply by A in above, $A(AX) = A(\lambda X)$

$$A^2 X = \lambda (AX) = \lambda (\lambda X) = \lambda^2 X$$

Hence λ^2 is an eigen value of A^2 with X itself as the corresponding eigen vector.

The theorem is true for n = 2

Let the result is true for n = k i.e, $A^k X = \lambda^k X$

premultiply by A in above, then $A(A^k X) = A(\lambda^k X)$

$$\Rightarrow \quad A^{k+1} X = \lambda^{k+1} X$$

Hence $\lambda^{k+1}$is an eigen value of $A^{k+1}$with X itself as the corresponding eigen vector.

Hence by mathematical induction, the theorem is true for all positive intergers n.

Theorem3: A square matrix A and its transpose A^T have the same eigen values.

Proof: we have $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$

$$|(A - \lambda I)^T| = |A^T - \lambda I|$$

$$|A - \lambda I| = |A^T - \lambda I|$$

Therefore, $|A - \lambda I| = 0$ if and only if $|A^T - \lambda I| = \mathbf{0}$

Thus the eigen values of A and A^T are same.

Theorem4: if λ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector X itself.

Proof: since A is non- singular and product of the eigen values is equal to $|A|$, so none of the eigen values of A is 0.

Since λ is an eigen value of A corresponding to the eigen vector X,

Then AX = λX

premultiply by A^{-1} in above, A^{-1} (AX) = A^{-1} (λX)

$$I X = \lambda (A^{-1} X) = \lambda (A^{-1} X)$$

$$\lambda^{-1} X = A^{-1} X, \text{ where } \lambda \neq 0$$

Hence by definition, then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector X

Theorem5: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: let A = $\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ 0 & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & a_{nn} \end{bmatrix}$

Given A is a triangular matrix.

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & . & . & . & a_{1n} \\ 0 & a_{22}-\lambda & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) = 0$

$\Rightarrow$ $\lambda = a_{11}, a_{22}, \dots, a_{nn}$

The Eigen values of A are $a_{11}, a_{22}, \dots, a_{nn}$ just the diagonal elements of the matrix.
The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Algebraic and geometric multiplicity of a characteristic root:

Def: suppose A is square matrix. If λ is a characteristic root of order t of the characteristic equation of A, then t is called the algebraic multiplicity of λ

Def: if s is the number of linearly independent characteristic vectors corresponding to the characteristic vector λ, then s is called the geometric multiplicity of λ.

Note: s ≤ t

Diagonalization of a matrix:

Def: A matrix A is diagonalizable if there exists and invariable matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. Also the matrix P is then said to diagonalize A or transform A to diagonal form.

Modal and Spectral matrices:

Def: The matrix P in $P^{-1}AP = D$ which diagonalize the square matrix A is called the Modal matrix of A and the resulting diagonal matrix D is known as Spectral matrix.

Note: 1. If the eigen values of A are all distinct, then it has n linearly independent eigen vectors and so it is diagonalizable.

2. Suppose A is a real symmetric matrix with n pair wise distinct eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the corresponding eigen vectors $X_1, X_2, \dots, X_n$ are pair wise orthogonal.

Hence if P = $(e_1, e_2, \dots, e_n)$, where $e_1 = \frac{X_1}{\|X_1\|}, e_2 = \frac{X_2}{\|X_2\|}, \dots, e_n = \frac{X_n}{\|X_n\|}$ then P is a orthogonal matrix.

Calculation of powers of a matrix:

We have $D = P^{-1}AP$

Then $D^2 = (P^{-1}AP)(P^{-1}AP)$

=> $= P^{-1}A^2P$

Similarly, $D^3 = P^{-1}A^3P$

In general, $D^n = P^{-1}A^nP$

Pre-multiply by P and post-multiply by P^{-1} then

we get $A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \lambda_n^n \end{bmatrix} P^{-1}$

Ex1:	If A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ then diagonalize the A and also find A^8
Sol :	Let A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

The eigen values of A are $1, 2, 3$

Case(i): put $\lambda = 1$

We have $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1 \leftrightarrow R_3 \begin{bmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2 \begin{bmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

This is an Echelon form.

Rank of A = 2 and number of variables = 3

Therefore, the system of infinite number of non-zero solutions

The number of arbitrary constants are n – r = 3 – 2 = 1

The equations are -4x - 4y + 2z = 0, y + z = 0

Let z = k, then y =$-k$, x = -k / 2

The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k/2 \\ -k \\ k \end{bmatrix} = \frac{-k}{2} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

The $X_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is the eigen vector corresponding to the eigen value λ = 1.

Case(ii): put $\lambda = 2$

We have $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \rightarrow R_3 - 4R_1 \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

	$R_3 \rightarrow R_3 + 3R_2 \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ This is an Echelon form. Rank of A = 2 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 2 = 1 The equations are - x + y + z = 0, z = 0 Let y = k, then x = k The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ The $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to the eigen value λ = 2 **Case(iii):** put $\lambda = 3$ We have $\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_3 \rightarrow R_3 - 2R_1 \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_3 \rightarrow R_3 + 2R_2 \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ This is an Echelon form. Rank of A = 2 and number of variables = 3 Therefore, the system of infinite number of non-zero solutions The number of arbitrary constants are n – r = 3 – 2 = 1 The equations are -2x + y + z = 0, - y + z = 0 Let z = k, then y = k , x = k The solution X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

	The $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = 3$ Let $P = [\ X_1, X_2, X_3\ ] =$ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$ Then consider $P^{-1} A P =$ $\begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$ Therefore A is diagonalizable. **Find A^8:** $A^8 = P\, D^8\, P^{-1} =$ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$

The following problems are discussed in the class work:

1	Diagonalize the matrix $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$
2	Diagonalize the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and also find A^4

The Cayley-Hamilton theorem:

Every square matrix satisfies its own characteristic equation.

Proof: Let $p(\lambda) = p_0 + p_1\lambda + \ldots + p_{n-1}\lambda^{n-1} + p_n\lambda^n$.

Let $\mathbf{B}(\lambda)$ be the adjugate matrix of the square matrix $\mathbf{A} - \lambda\mathbf{I}$, which may be considered as a polynomial in λ and with matrix coefficients

$\mathbf{B}(\lambda) = \mathbf{B}_0 + \lambda\mathbf{B}_1 + \ldots + \lambda^{q-1}\mathbf{B}_{q-1} + \lambda^q\mathbf{B}_q$, where $\mathbf{B}_q$ are constant matrices.

By the formula $(\mathrm{adj}\mathbf{A})\,\mathbf{A} = (\det\mathbf{A})\,\mathbf{I}$, we have

$\mathbf{B}(\lambda)(\mathbf{A} - \lambda\mathbf{I}) = p(\lambda)\mathbf{I} = p_0\mathbf{I} + p_1\lambda\mathbf{I} + \ldots + p_{n-1}\lambda^{n-1}\mathbf{I} + p_n\lambda^n\mathbf{I}.$

On the other hand, we have

$\mathbf{B}(\lambda)(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{B}_0\mathbf{A} + \lambda(\mathbf{B}_1\mathbf{A} - \mathbf{B}_0) + ... + \lambda^q(\mathbf{B}_q\mathbf{A} - \mathbf{B}_{q-1}) - \lambda^{q+1}\mathbf{B}_q$.

Thus we get q = n -1 and

$\mathbf{B}_0\mathbf{A} = p_0\mathbf{I}$,
$\mathbf{B}_1\mathbf{A} - \mathbf{B}_0 = p_1\mathbf{I}$,
:
$\mathbf{B}_{n-1}\mathbf{A} - \mathbf{B}_{n-2} = p_{n-1}\mathbf{I}$,
$- \mathbf{B}_{n-1} = p_n\mathbf{I}$.

Multiplying powers of **A** on the right sides, we get

$\mathbf{B}_0\mathbf{A} = p_0\mathbf{I}$,
$\mathbf{B}_1\mathbf{A}^2 - \mathbf{B}_0\mathbf{A} = p_1\mathbf{A}$,
:
$\mathbf{B}_{n-1}\mathbf{A}^n - \mathbf{B}_{n-2}\mathbf{A}^{n-1} = p_{n-1}\mathbf{A}^{n-1}$,
$- \mathbf{B}_{n-1}\mathbf{A}^n = p_n\mathbf{A}^n$.

Adding all the equalities together, we get

$p(\mathbf{A}) = p_0\mathbf{I} + p_1\mathbf{A} + ... + p_{n-1}\mathbf{A}^{n-1} + p_n\mathbf{A}^n = \mathbf{O}$.

Applications of Cayley – Hamilton theorem:

1. To find the inverse of a matrix
2. To find higher powers of the matrix

Ex1:	Verify Cayley Hamilton theorem and find the inverse of $\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$
Sol:	The characteristic equation of A is $\lvert A - \lambda I \rvert = 0$ $\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2 \end{vmatrix} = 0$ $\Rightarrow (\lambda^3 - 3\lambda^2 - 7\lambda - 1) = 0$ Consider $A^3 - 3A^2 - 7A - I$ $= \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} + \begin{bmatrix} -21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix} + \begin{bmatrix} -14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Cayley Hamilton theorem is verified. Consider $A^3 - 3A^2 - 7A - I = 0$

	$A^{-1}(A^3 - 3A^2 - 7A - I) = 0$ $A^{-1} = A^2 - 3A - 7I = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$
Ex2:	If A = $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, find the value of the matrix $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$
Sol:	The characteristic equation of A is $\|A - \lambda I\| = 0$ $\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0$ $\Rightarrow (\lambda^3 - 5\lambda^2 + 7\lambda - 3) = 0$ By Cayley – Hamilton theorem, we have $\Rightarrow (A^3 - 5A^2 + 7A - 3I) = 0$ Consider, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ $= A^5(A^3\text{-}5A^2 + 7A - 3I)$+A$(A^3\text{-}5A^2 + 8A - 2I)$ + I = A[$(A^3\text{-}5A^2 + 7A - 3I)$ + (A+I)] + I = A$(A^3\text{-}5A^2 + 7A - 3I) + A^2 + A + I$ $= A^2 + A + I$ $= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

The following problems are discussed in the class work:

1	Verify Cayley- Hamilton theorem for the matrix and also find the inverse of the matrix $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$
2	Verify Cayley – Hamilton theorem for A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and also find A^4

Quadratic forms

A homogenous expression of the second degree in any number of variables is called a quadratic form.

Ex: $3x^2 + 5xy - 2y^2$ is a quadratic form in two variables x and y.

Def: An expression of the form Q = $X^T A X = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j, where\ a'_{ij}s$ are constants, is called a quadratic form in n variables $x_1, x_2, \dots, x_n$. If the constants $a'_{ij}s$ are real numbers it is called a real quadratic form.

i.e, $X^T A X = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j$

$$= \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Where X = $\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and A is known as the matrix of the quadratic form.

Matrix of Quadratic form: any quadratic form Q can be expressed as Q = $X^T A X$

The symmetric matrix A is called the matrix of the quadratic form Q and $|A|$ is called the discriminant of the quadratic form.

If $|A| = 0$, the quadratic form is called singular, otherwise non-singular. In other words, if the rank of A is r < n then the quadratic form is singular otherwise non-singular.
Consider the quadratic form $x^2 + 2y^2 + 7z^2 + 2xy + 6xz + 10yz$
Write the coefficients of square terms along the diagonal and divide the coefficients of the product terms xy, xz, yz by 2 and write them at the appropriate places.

	x	y	z
x	x2	xy/2	xz/2
y	yx/2	y2	yz/2
z	zx/2	zy/2	z2

Thus the matrix of the above quadratic form is

$$\begin{bmatrix} 1 & 2/2 & 6/2 \\ 2/2 & 2 & 10/2 \\ 6/2 & 10/2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & 5 & 7 \end{bmatrix}$$

Rank of the Quadratic form:

Let $X^T A\, X$ be a quadratic form. The rank r of A is called the rank of the quadratic form.

Canonical from (or) Normal form of a Quadratic form:

Let $X^T A\, X$ be a quadratic form in n variables. Then there exists a real non – singular linear transformation X = PY which transform $X^T A\, X$ to another quadratic form of type $Y^T D\, Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$, then $Y^T D\, Y$ is called the Canonical form of $X^T A\, X$. Here D = diag[$\lambda_1, \lambda_2, \ldots, \lambda_n$]

Def: Let Q = = $X^T A\, X$ be a quadratic form in n variables $x_1, x_2, \ldots, x_n$,

Index (s): The number of positive terms in its canonical form is called the index of the quadratic form.

Signature: signature of the quadratic form is the difference of positive and negative terms in the canonical form.

i.e, s – (r – s) = 2s – r is called the signature of the quadratic form.

Nature of the quadratic form: If the rank of the matrix A is r and the signature of the quadratic form Q is s, then the quadratic form is said to be

(i) **Positive definite:** if r = n and s = n (or) if all the eigen values of A > 0.

(ii) **Negative definite:** if r = n and s = 0 (or) if all the eigen values of A < 0.

(iii) **Positive semi definite:** if r < n and s = r (or) if all the eigen values of A > 0 and at least one eigen value = 0.

(iv) **Negative semi definite:** if r <n and s = 0 (or) if all the eigen values of A < 0 and at least one eigen value = 0.

(v) **Indefinite:** in all other cases (or) if some of the eigen values of A are positive and others negative.

Methods of Reduction of Quadratic form to Canonical form (or Sum of Squares form)

Any quadratic form may be reduced to canonical form by using following methods:

1. Diagonalization (Reduction to canonical form using Linear transformation)
2. Orthogonalisation (Reduction to canonical form using orthogonal transformation)
3. Lagrange's reduction.

Reduction to canonical form using Linear transformation (Diagonalisation):

Let $X^T A\, X$ be a quadratic form, where A is the matrix of the quadratic form.

Let X = PY be the non – singular linear transformation

Then we have $X^T A\, X = (PY)^T A\, (PY)$

$$= (Y^T P^T) A (PY)$$

$= Y^T(P^T AP)Y$

$= Y^T\ D\ Y$, where D = $P^T AP$

Here $Y^T\,D\,Y$ is called the canonical form of the quadratic form.
Congruent matrices: the matrices D and A are congruent matrices and the transformation X = PY is known as congruent transformation.

Working rule to reduce Quadratic form to canonical form:

Step1: write the symmetric matrix of the given quadratic form.

Step 2: write the matrix A in the following relation: $A_{n\times n} = I_n\ A\ I_n$.

Step 3: reduce the matrix A on left hand side to a diagonal matrix (i) by applying elementary row operations on the left identity matrix and on A on left hand side (ii) by applying elementary column operations on the right identity matrix and on A on left hand side.

Step 4: by these operations, A = IAI will be reduced to the form D = $P^T AP$

Where D is the diagonal matrix and P is the matrix used in the linear transformation.

The canonical form is given by

$$Y^T\,D\,Y = [y_1 \quad y_2 \quad \dots\ y_n]\begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

$$= d_1 y_1^2 + d_2 y_2^2 + \ \dots + \ d_n y_n^2$$

Ex1:	Reduce the following quadratic form to normal form and hence find its rank, index, signature and nature: $10x^2 + 2y^2 + 5z^2 + 6yz - 10zx - 4xy$
Sol:	The matrix of the Quadratic form is $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$ We write A = $I_3\ A\ I_3$ $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \to 5R_2 + R_1, R_3 \to 2R_3 + R_1,$ $\begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $C_2 \to 5C_2 + C_1, C_3 \to 2C_3 + C_1,$ $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

	$R_3 \to 2R_3 - R_2$, $\begin{bmatrix}10 & 0 & 0\\ 0 & 40 & 20\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 5 & 0\\ 1 & -5 & 4\end{bmatrix} A \begin{bmatrix}1 & 1 & 1\\ 0 & 5 & 0\\ 0 & 0 & 2\end{bmatrix}$ $C_3 \to 2C_3 - C_2$, $\begin{bmatrix}10 & 0 & 0\\ 0 & 40 & 20\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 5 & 0\\ 1 & -5 & 4\end{bmatrix} A \begin{bmatrix}1 & 1 & 1\\ 0 & 5 & -5\\ 0 & 0 & 4\end{bmatrix}$ This is of the form D = P^T A P, where D = $\begin{bmatrix}10 & 0 & 0\\ 0 & 40 & 0\\ 0 & 0 & 0\end{bmatrix}$ and P = $\begin{bmatrix}1 & 1 & 1\\ 0 & 5 & -5\\ 0 & 0 & 4\end{bmatrix}$ The linear transformation is X = PY i.e, $\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 0 & 5 & -5\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$ The quadratic form can be to the canonical form is Y^T A Y = 10 y_1^2 + 40 y_2^2 Rank = 2 Index = 2 Signature = 2 (2) – 2 = 2 Nature = Positive semi-definite.
Ex2:	Reduce the quadratic form to the canonical form $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$
Sol:	The matrix of the Quadratic form is $\begin{bmatrix}3 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix}$ We write A = I_3 A I_3 $\begin{bmatrix}3 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ $R_2 \to 3R_2 + R_1, C_2 \to 3C_2 + C_1$, $\begin{bmatrix}3 & 0 & 0\\ 0 & 5 & -3\\ 0 & -3 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 1 & 0\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}$ $R_3 \to 5R_3 + 3R_2, C_3 \to 5C_3 + 3C_2$, $\begin{bmatrix}3 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 3 & 0\\ 3 & 9 & 5\end{bmatrix} A \begin{bmatrix}1 & 1 & 3\\ 0 & 3 & 9\\ 0 & 0 & 5\end{bmatrix}$

	$\frac{R_1}{\sqrt{3}}, \frac{C_1}{\sqrt{3}}, \frac{R_2}{\sqrt{5}}, \frac{C_2}{\sqrt{5}}, \frac{R_3}{\sqrt{6}}, \frac{C_3}{\sqrt{6}}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{5}} & 0 \\ \frac{3}{\sqrt{6}} & \frac{9}{\sqrt{6}} & \frac{5}{\sqrt{6}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{5}} & \frac{9}{\sqrt{6}} \\ 0 & 0 & \frac{5}{\sqrt{6}} \end{bmatrix}$ This is of the form D = P^T A P The canonical form is $y_1^2 + y_2^2 + y_3^2$ The linear transformation is X = PY i.e, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{5}} & \frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{5}} & \frac{9}{\sqrt{6}} \\ 0 & 0 & \frac{5}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

The following problems are discussed in the class work:

1	Reduce the quadratic form to the canonical form $x^2 + y^2 + 2z^2 - 2xy + 4zx + 4yz$
2	Reduce the following quadratic form to normal form and hence find its rank, index, signature and nature: $10x^2 + 2y^2 + 5z^2 - 4xy - 10zx + 6yz$

Reduction to normal form by orthogonal transformation:

In the transformation X = PY, P is an orthogonal matrix and if X = PY transforms the quadratic form Q to the canonical form then Q is said to be reduced to the canonical form by an orthogonal transformation.

Suppose A has Eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct) and $X_1, X_2, \ldots, X_n$ are three eigen vectors which are linearly independent, we can construct normalized eigen vectors e_1, e_2, e_3 corresponding to $\lambda_1, \lambda_2, \lambda_3$ which are pair wise orthogonal. Then we define P = (e_1, e_2, e_3), where $e_1 = \frac{X_1}{\|X_1\|}$, $e_2 = \frac{X_2}{\|X_2\|}$, $e_3 = \frac{X_3}{\|X_3\|}$

The Mean Value Theorems are some of the most important theoretical tools in Calculus and they are classified into various types. In these free GATE Study Notes, we will learn about the important Mean Value Theorems like Rolle's Theorem, Lagrange's Mean Value Theorem, Cauchy's Mean Value Theorem and Taylor's Theorem.

Rolle's Theorem

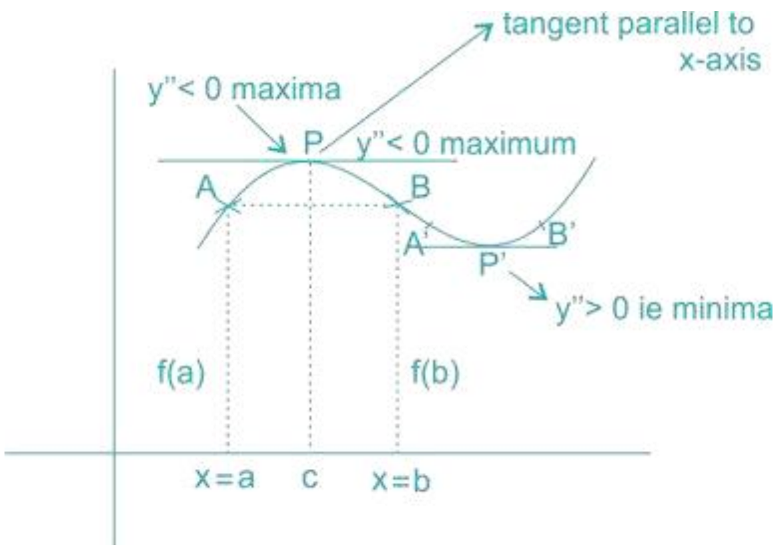
Statement: If a real valued function f(x) is

1. Continuous on [a,b]
2. Derivable on (a,b) and f(a) = f(b)

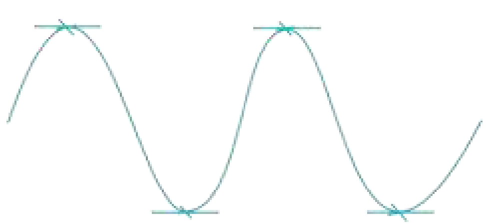
Then there exists at least one value of x say c ϵ (a,b) such that f'(c) = 0.

1. Geometrically, Rolle's Theorem gives the tangent is parallel to x-axis.

Rolle's Theorem gives the tangent is parallel to x-axis.



3. For a continuous curve maxima and minima exists alternatively



4. Geometrically y'' gives concaveness i.e.

i. y'' < 0 ⇒ Concave downwards and indicates maxima.

ii. y'' > 0 ⇒ Concave upwards and indicates minima.

To know the maxima and minima of the function of single variable Rolle's Theorem is useful.

5. y''=0 at the point is called point of inflection where the tangent cross the curve is
4. called point of inflection and
6. Rolle's Theorem is fundamental theorem for all Different Mean Value Theorems

Q1) The function is given as f(x) = (x–1)2(x–2)3 and x ϵ [1,2]. By Rolle's Theorem find the value of c ?

Sol. Given f(x) = (x–1)2(x–2)3 f(x) is continuous on [1,2] i.e. f(x) = finite on [1,2] f'(x) = 2(x–1)(x–2)3 + 3(x–1)2(x–2)2
f'(x)is finite in (1,2) hence differentiable then c ∈ (1,2)
∴f'(c) = 0
2(c–1)(c–2)3 + 3(c–1)2(c–2)2 = 0
(c-1)(c-2)2[2c – 4 + 3c – 3] = 0
(c–1)(c–2)2[5c–7] = 0
$\therefore c = \frac{7}{5} = 1.4 \in (1,2)$

Q2) Discuss the applicability of Rolle's theorem to the function f(x) = $\frac{1}{x^2}$ in [-1, 1]

Ans: Given function f(x) = $\frac{1}{x^2}$ in [-1, 1]

clearly f(x) is not defined at 0 in [-1,1]

so, f(x) is not continuous at x=0

∴ f(x) is not continuous in [-1,1]

∴ function f(x) fails first condition of Rolle's theorem

So Rolle's theorem is not applicable for this function

Q3) Find c of Rolle's theorem for the function f(x)=x^2 in $[-1,1]$

Ans:Given functions $f(x) = x^2 \ \ in \ [-1,1], f'(x) = 2x$

By Rolle's theorem $f'(c) = 0$

⇒ 2c=0

$c = 0$ in [-1,1].

Q4) Is the Rolle's theorem applicable to the function f(x) = x^2 in [1, 2]?

Ans:Given function $f(x) = x^2 \ in \ [1,2]$

Clearly function f(x) is polynomial of degree 2,$\lim_{x \to a} x^2 = a^2$

∴ f(x) is continuous in [1,2].Clearly $f'(x)$ is exist in [1,2]

But $f(1) = 1, f(2) = 4, f(1) \neq f(2)$

Function f(x) is not satisfy the third condition of Rolle's theorem

∴ Rolle's theorem is not applicable for this function.

Q5) State Rolle's theorem

Ans: If function f(x) is defined on [a,b] such that

(i) f is continuous on [a,b]

(ii)f is differentiable on (a,b)

(iii) f(a)=f(b)

Then there exist at least one point c$\in$ (a,b) such that $f'(c)=0$.

Q6) Verify Rolles's theorem for the function $f(x)=\frac{sinx}{e^x}$ in the interval [0,π].

Solution: (i) $sinx$ and e^x both are continuous functions in[0,π]

therefore $f(x)=\frac{sinx}{e^x}$ is also continuous on [0,π].

(ii) Also, since $sinx$ and e^x are derivable in (0,π) then $\frac{sinx}{e^x}$ is also derivable in (0,π).

(iii) $f(0)=\frac{sin0}{e^0}=\frac{0}{1}=1,\ f(p)=\ \frac{sinp}{e^p}=\frac{0}{e^p}=0.$

Thus all 3 condition's of Rolle's theorem are satisfied.

There exist $c?\,(0,\mathrm{p})$ such that $f^{|}(c)=0$.

We have $f(x)=\frac{sinx}{e^x}$

$$f^{|}(x)=\frac{e^x(sinx)^{|}-sinx\ (e^x)^{|}}{(e^x)^2}$$

$$=\frac{e^x\,(\cos x-\sin x)}{(e^x)^2}$$

$$=\frac{\cos x-\sin x}{e^x}$$

$$f^{|}(c)\quad=\frac{\cos c-\sin c}{e^c}$$

We have $f^{|}(c)=0$

$\Rightarrow \cos c-\sin c=0$

$\Rightarrow \cos c=\sin c$

$\Rightarrow \frac{\sin c}{\cos c} = 1$

$\Rightarrow \tan c = 1$

$\Rightarrow \tan c = tan\frac{p}{4}$

$\Rightarrow c = \frac{p}{4}$?(0,π).

Hence Rolle's theorem is verified.

Q) Discuss the applicability of Rolle's theorem to the function $f(x) = 2 + (x-1)^{\frac{2}{3}}$ in the interval [0,2].

Solution: (i) $f(x)$ is continuous on [0,2].

(ii) $f^{|}(x) = 0 + \frac{2}{3}(x-1)^{\frac{2}{3}-1}$

$= \frac{2}{3}\frac{1}{(x-1)^{\frac{1}{3}}}$

Thus $f^{|}(x)$ does not exist in [0,2] at $x = 1$.

Therefore $f(x)$ does not satisfy the condition of Rolle's theorem on [0,2].

Hence Rolle's theorem is not applicable.

Lagrange's Mean Value Theorem

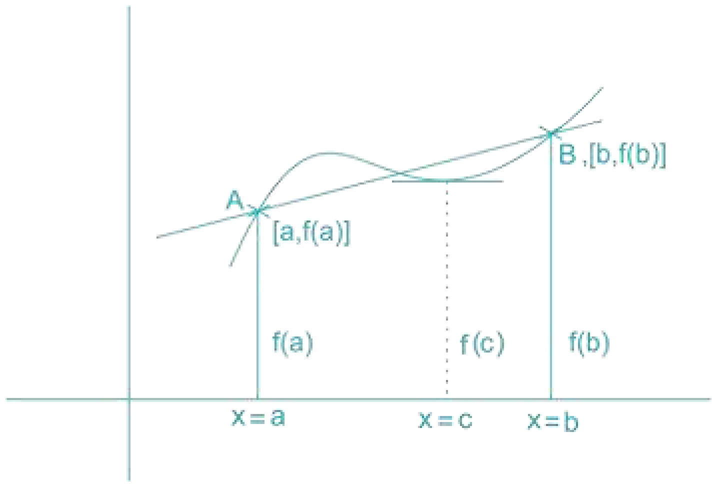
Statement: If a Real valued function f(x) is

1. Continuous on [a,b]

2. Derivable on (a,b)

Then there exists at least one value $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Geometrically, slope of chord AB = slope of tangent



Application:
1. To know the approximation of algebraic equation, trigonometric equations etc.
2. To know whether the function is increasing (or) decreasing in the given interval.

Q7) Find the value of c is by using Lagrange's Mean Value Theorem of the function
$f(x) = x(x-1)(x-2)\ x \in \left[0, \frac{1}{2}\right]$

Sol: f(x) is continuous in [0, 1/2] and it is differentiable in (0, 1/2) f'(x)
= (x2 – x)[1] + (x – 2)(2x – 1)
= x2 – x + 2x2 – x – 4x + 2 = $3x^2 - 6x + 2$
From Lagrange's Mean Value Theorem we have, $f'(c) = 3c^2 - 6c + 2 =$
f (1/2) – f (0)/1/2=3/4
$12c^2 - 24c + 8 - 3 = 0$
12 c 2 – 24 c + 5 = 0

$\Rightarrow c = \frac{24 \pm \sqrt{576-240}}{24}$

$c = 1 \pm \frac{\sqrt{21}}{6}$

$\therefore c = 1 - \frac{\sqrt{21}}{6} \in \left(0, \frac{1}{2}\right)$

Q8) Find c of Lagrange's mean value theorem for f(x)=$log_e x\ in(1,e)$

Ans: Given functions $f(x) = \log_e x\ in\ (1, e), f'(x) = \frac{1}{x}$

By Lagrange's mean value theorem,$f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow \frac{1}{c} = \frac{f(e) - f(1)}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{1 - 0}{e - 1}$$

$c = e - 1$ in $(1, e)$

Q9) Find the value of 'c' of Lagrange's mean value theorem for the function f(x) = x^2 in [1, 5].

Ans: Given functions $f(x) = x^2\ in\ [1,5], f'(x) = 2x$

By Lagrange's mean value theorem,$f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow 2c = \frac{f(5)-f(1)}{5-1}$$

$$\Rightarrow 2c = \frac{25-1}{4}$$

$c = 3$ in $(1,5)$.

Q10) State Lagrange's mean value theorem.

Ans: If function f(x) is defined on [a,b] such that

(i) f is continuous on [a,b]

(ii)f is differentiable on (a,b)

Then there exist at least one point c∈ (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Q11) What is the Lagrange's remainder for Taylor's theorem

Ans: $f(b) = f(a) + (b-a)f'(a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f'''(a)}{3!}(b-a)^3 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(b-a)^{n-1} + R_n$where

Lagrange's remainder,$R_n = \frac{(b-a)^n f^n(c)}{n!}$

Q12) Apply Mean value Theorem to show that $\frac{b-a}{\sqrt{1-a^2}} < sin^{-1}b - sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$ where $0 < a < b < 1$.

Solution: Let $f(x) = sin^{-1}x$, $sin^{-1}x$ is continuous and differentiable in [a,b].

$f^{|}(x) = \frac{1}{\sqrt{1-x^2}}$

By Lagrange's mean value theorem, there exist c? (a, b) such that $f^{|}(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{\sqrt{1-c^2}} = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{sin^{-1}b - sin^{-1}a}{b-a} \quad ----- \quad (1)$$

We have c? (a, b)

$\Rightarrow a < c < b$

$\Rightarrow a^2 < c^2 < b^2$

$\Rightarrow -a^2 > -c^2 > -b^2$

$\Rightarrow 1 - a^2 > 1 - c^2 > 1 - b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{sin^{-1}b - sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < sin^{-1}b - sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$.

Q13) Verify Lagrange's mean value Theorem for the function $f(x) = x(x-1)(x-2)$ in the interval [0,1/2].

Solution : (i) $f(x)$ is continuous on [0,1/2].

(ii) $f(x)$ is derivable on (0,1/2).

$f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$

$f^{|}(x) = 3x^2 - 6x + 2$

By (i) and (ii) $f(x)$ satisfies Lagrange's Mean value Theorem.

There exist c? $(0, \frac{1}{2})$ such that $f^{|}(c) = \frac{f(b)-f(a)}{b-a}$

$\Rightarrow 3c^2 - 6c + 2 =$

$\Rightarrow 3c^2 - 6c + 2 =$

$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$

$\Rightarrow 12c^2 - 24c + 8 - 3 = 0$

$\Rightarrow 12c^2 - 24c + 5 = 0$

$$\Rightarrow c = \frac{-(-24) \pm \sqrt{24^2 - 4(12)(5)}}{2(12)}$$

$$= \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$= \frac{24 \pm \sqrt{336}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24}$$

$$= 1 \pm \frac{\sqrt{21}}{6}$$

$$= 1 \pm 0.7637$$

= 1.7637 or 0.2363

= 0.2363

Therefore c? $(0, \frac{1}{2})$.

Therefore Lagrange's Mean Value Theorem is verified.

Cauchy's Mean Value Theorem
Statement: If two functions f(x) and g(x) are
1. Continuous on [a,b]
2. Differentiable on (a,b) and $g'(x) \neq 0$ then there exists at least one value of x such

that $c \in (a,b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Generally, Lagrange's mean value theorem is the particular case of Cauchy's mean value theorem.

Q14)If f(x) = ex and g(x) = e-x, xϵ[a,b]. Then by the Cauchy's Mean Value Theorem the value of c ?

Sol. Here both f (x) = e x and g (x) = e - x are continuous on [a,b] and differentiable in (a,b)
From Cauchy's Mean Value theorem,
f ′ (c)/g ′ (c) =f (b) − f (a)/g (b) − g (a)

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$e^{2c} = e^{a+b} \Rightarrow \quad c = \frac{a+b}{2}$$

Therefore, c is the arithmetic mean of a and b.

Q15) State Cauchy's mean value theorem

Ans: Statement: If functions f(x) and g(x) are defined on [a,b] such that

(i)f, g are continuous on [a,b]

(ii)f,g are differentiable on (a,b) and

(iii) $g'(x) \neq 0 \ \forall\ x \in (a, b)$

Then there exist a point c$\in (a,b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Q16) Find the value of 'c' of Cauchy's mean value theorem for the function f(x) = e^x and g(x) = e^{-x} in [a, b].

Ans:

Given functions $f(x) = e^x, g(x) = e^{-x}\ in\ [a, b]$

$$f'(x) = e^x, g'(x) = -e^{-x}$$

By Cauchy's mean value theorem

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\Rightarrow -e^{2c} = e^b.e^a\frac{e^b - e^a}{e^a - e^b}$$

$$\Rightarrow e^{2c} = e^{a+b}$$

$c = \frac{a+b}{2}$ in [a,b]

Q17) Find c of Cauchy's mean value theorem for $f(x)=x^2, g(x)=x^3\ in[1,2]$

Ans: Given functions $f(x) = x^2, g(x) = x^3\ in\ [1,2], f'(x) = 2x, g'(x) = 3x^2$

By Cauchy's mean value theorem ,$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow \frac{2c}{3c^2} = \frac{f(2) - f(1)}{g(2) - g(1)}$$

$$\Rightarrow \frac{2c}{3c^2} = \frac{4 - 1}{8 - 1}$$

$$\Rightarrow \frac{2}{3c} = \frac{3}{7}$$

$$c = \frac{14}{9}$$

Taylor's Theorem

It is also called as higher order mean value theorem.

Statement: If fn(x) is

1. Continuous on [a, a + x] where x = b – a

2. Derivable on (a, a + x)

Then there exists at least one number θ (0,1) (1-θ ≠ 0) such that,

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n$ ___(1)

Where R_n = Lagrange's form of remainder $= \frac{h^n}{n!}f^n(a+\theta h)$

Also Cauchy's form of remainder $R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$

Note:

Substituting a = 0 and h = x in equation (1) (Taylor's series equation) we get,

$f(x) = f(0) + x f'(0) + x^2/2!\, f''(0) + x^3/3!\, f'''(0) + \cdots x^{n-1}/(n-1)!\, f n-1(0) + R n$

This is known as Maclaurin's series.

Here $R n = x^n/n!\, f n(\theta x)$ is called Lagrange ' s form of remainder,

$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x)$ is called Cauchy's form of remainder

Q18) Expand e^x using Taylor's series up to second degree terms about x=1

Ans: Let f(x)= e^x

By Taylor's series about x=1

$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots$ ------(1)

Since f(x)=e^x ; f(1)= $e, f'(x) = e^x$; $f'(1) = e; f''(x) = e^x$; $f''(1) = e$

Put these values in (1) we get ,$e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \cdots$

Q19) Obtain the Taylor's series expansion of the function f(x) = sinx up to third degree term about the point x =$\frac{\pi}{4}$

Ans:Let f(x)= $sinx$

By Taylor's series about x=$\frac{\pi}{4}$

$f(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3 + \cdots$ ------(1)

Since f(x)=sinx ; f($\frac{\pi}{4}$)=$\frac{1}{\sqrt{2}}$;$f'(x) = cosx$; $f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$;$f''(x) = -sinx$; $f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$f'''(x) = -cosx\,; f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Put these values in (1) we get ,$sinx = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \cdots$

Q20) Obtain the Maclaurin's series expansion term of the function f(x) = e^x up to third degree

Ans:Let f(x)= e^x

By Maclaurin series,$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$ ------(1)

Since f(x)=e^x ; f(0)=1;$f'(x) = e^x$;$f'(0) = 1$;$f''(x) = e^x$;$f''(0) = 1$;$f'''(x) = e^x$;$f''(0) = 1$

Put these values in (1) we get ,$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Q21) Expand sinx using Maclaurin's series upto second degree terms

Ans: Let f(x)= sinx

By Maclaurin series ,$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$ ------(1)

Since f(x)=sinx ; f(0)=0

$$f'(x) = cosx \;\; ; f'(0) = 1$$

$$f''(x) = -sinx \;\; ; f''(0) = 0$$

$$f''{'(x)} = -cosx \;\; ; f''(0) = -1$$

Put these values in (1) we get, $sinx = x - \frac{x^3}{3!} + \cdots$

Q22) Expand cosx using Maclaurin's series upto second degree terms

Ans: Let f(x)= cosx

By Maclaurin's series

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$ ------(1)

Since f(x)=cosx ; f(0)=1

$$f'(x) = -sinx \;\; ; f'(0) = 0$$

$$f''(x) = -cosx \;\; ; f''(0) = -1$$

Put these values in (1) we get , $cosx = 1 - \frac{x^2}{2!} + \cdots$

UNIT- IV

PARTIAL DIFFERENTIATION

Total derivative: **If** $u = f(x, y)$ **where** $x = \varphi(t)$ **and** $y = \psi(t)$**, then we can express** u **as a function of** t **alone by substituting the values of** x **and** y **in** $f(x, y)$**. Thus we can find the ordinary derivative** $\frac{du}{dt}$ **which is called the** total derivative **of** u **to distinguish it from partial derivatives** $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$**.**

To find $\frac{du}{dt}$ without substituting the values of x and y, we establish the **chain rule.**

$$\boldsymbol{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

Note: If $u = f(x, y, z)$ where $x, y\ and\ z$ are all functions of t, then chain rule is

$$\boldsymbol{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}}$$

Differentiation of implicit functions: If $f(x, y) = c$ be an implicit relation between $x\ and\ y$ which
defines as a differentiable function of x, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \Longrightarrow 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \Longrightarrow \boldsymbol{\frac{dy}{dx}} = -\boldsymbol{\frac{\partial f}{\partial x}}/\boldsymbol{\frac{\partial f}{\partial y}}$$

Problem: If the curves $f(x, y) = 0$ and $\emptyset(y, z) = 0$ touch then show that $f_y \emptyset_z \frac{dz}{dx} = f_x \emptyset_y$.

Sol: $f(x, y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$ and $\emptyset(y, z) = 0 \Rightarrow \frac{dy}{dz} = -\frac{\emptyset_z}{\emptyset_y}$

Consider $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = -\frac{\emptyset_y}{\emptyset_z} \cdot -\frac{f_x}{f_y} = \frac{\emptyset_y}{\emptyset_z} \cdot \frac{f_x}{f_y} \Rightarrow\ f_y \emptyset_z \frac{dz}{dx} = f_x \emptyset_y$

Problem: If $u = tan^{-1}\left(\frac{y}{x}\right)$ where $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$, find $\frac{du}{dt}$

Sol: Given $u = tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{x^2}{x^2+y^2} \cdot -\frac{y}{x^2} \cdot (e^t + e^{-t}) + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} \cdot (e^t - e^{-t})$$

$$= \frac{-y}{x^2+y^2} \cdot y + \frac{x}{x^2+y^2} \cdot x$$

$$= \frac{x^2-y^2}{x^2+y^2} = -\frac{2}{e^{2t}+e^{-2t}}$$

Problem: If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$

Sol: Given $x^3 + y^3 + 3xy = 1$.

Suppose $f(x, y) = x^3 + y^3 + 3xy - 1$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2+3y}{3y^2+3x} = -\frac{x^2+y}{y^2+x}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \left(1.\log xy + x.\frac{1}{xy}.y\right) + x.\frac{1}{xy}.x.\frac{dy}{dx}$$

$$= \log xy + 1 + \frac{x}{y}.\left(-\frac{x^2+y}{y^2+x}\right)$$

$$= 1 + \log xy - \frac{x(x^2+y)}{y(y^2+x)}$$

Problem: If $U = F\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then prove that $x^2\frac{\partial U}{\partial x} + y^2\frac{\partial U}{\partial y} + z^2\frac{\partial U}{\partial z} = 0$.

Sol: Given $U = F\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

Suppose $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$, $s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$, then $U = F(r, s)$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial U}{\partial r}.\frac{-1}{x^2} + \frac{\partial U}{\partial s}.\frac{-1}{x^2} \implies x^2\frac{\partial U}{\partial x} = -\frac{\partial U}{\partial r} - \frac{\partial U}{\partial s}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial U}{\partial r}.\frac{1}{y^2} + \frac{\partial U}{\partial s}.0 \implies y^2\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r}$$

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial U}{\partial r}.0 + \frac{\partial U}{\partial s}.\frac{1}{z^2} \implies z^2\frac{\partial U}{\partial z} = \frac{\partial U}{\partial s}$$

Therefore $x^2\frac{\partial U}{\partial x} + y^2\frac{\partial U}{\partial y} + z^2\frac{\partial U}{\partial z} = -\frac{\partial U}{\partial r} - \frac{\partial U}{\partial t} + \frac{\partial U}{\partial r} + \frac{\partial U}{\partial s} = 0$.

Problem: If $z = f(x, y)$, $x = e^u + e^{-v}$ and $y = e^u - e^{-v}$, then prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Sol: Given$z = f(x, y)$ and $x = e^u + e^{-v}$, $y = e^u - e^{-v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}e^u$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}.-e^{-v} + \frac{\partial z}{\partial y}e^{-v}$$

Consider $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}e^u\right) - \left(\frac{\partial z}{\partial x}.-e^{-v} + \frac{\partial z}{\partial y}e^{-v}\right)$

$$= \frac{\partial z}{\partial x}(e^u + e^{-v}) + \frac{\partial z}{\partial y}(e^u - e^{-v})$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Problem: If by the substitution $u = x^2 - y^2, v = 2xy, f(x,y) = \theta(u,v)$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

Sol: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_u.2x + f_v.2y = 2(xf_u + yf_v)$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = f_u.-2y + f_v.2x = 2(-yf_u + xf_v)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2(xf_u + yf_v)\right)$$

$$= 2\frac{\partial}{\partial x}(xf_u) + 2\frac{\partial}{\partial x}(yf_v)$$

$$= 2\left[x\left(f_{uu}\frac{\partial u}{\partial x} + f_{uv}\frac{\partial v}{\partial x}\right) + f_u.1\right] + 2y\left(f_{uv}\frac{\partial u}{\partial x} + f_{vv}\frac{\partial v}{\partial x}\right)$$

$$= 2[x(f_{uu}.2x + f_{uv}.2y) + f_u.1] + 2y(f_{uv}.2x + f_{vv}.2y)$$

$$= 4x^2 f_{uu} + 4y^2 f_{vv} + 8xyf_{uv} + 2f_u$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2(-yf_u + xf_v)\right)$$

$$= -2\frac{\partial}{\partial y}(yf_u) + 2\frac{\partial}{\partial y}(xf_v)$$

$$= -2\left[y\left(f_{uu}\frac{\partial u}{\partial y} + f_{uv}\frac{\partial v}{\partial y}\right) + f_u.1\right] + 2x\left(f_{uv}\frac{\partial u}{\partial y} + f_{vv}\frac{\partial v}{\partial y}\right)$$

$$= -2[y(f_{uu}.-2y + f_{uv}.2x) + f_u.1] + 2x(f_{uv}.-2y + f_{vv}.2x)$$

$$= 4y^2 f_{uu} + 4x^2 f_{vv} - 8xyf_{uv} - 2f_u$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (4x^2 f_{uu} + 4y^2 f_{vv} + 8xyf_{uv} + 2f_u) + (4y^2 f_{uu} + 4x^2 f_{vv} - 8xyf_{uv} - 2f_u)$$

$$= 4(x^2 + y^2)(f_{uu} + f_{vv})$$

$$= 4(x^2 + y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

Problem: Transform $\frac{\partial^2 z}{\partial x^2} + 2xy^2\frac{\partial z}{\partial x} + 2(y - y^3)\frac{\partial z}{\partial y} + x^2y^2 z = 0$ by the substitution $x = uv, y = \frac{1}{v}$.

Hence show that z is the same function of u v and as of x and y.

Sol: Given that $x = uv, y = \frac{1}{v} \Rightarrow u = xy$ and $v = \frac{1}{y}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = z_u y + z_v.0 = yz_u$$

$$\Longrightarrow z_x = yz_u$$

$$\Longrightarrow xz_x = xyz_u = uz_u$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = z_u x + z_v.-\frac{1}{y^2}$$

$$\Longrightarrow yz_y = xyz_u - \frac{1}{y}z_v$$

$$= uz_u - vz_v$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(yz_u) = y\left(z_{uu}\frac{\partial u}{\partial x} + z_{uv}\frac{\partial v}{\partial x}\right)$$

$$= y(z_{uu}.y + z_{uv}.0)$$

$$= y^2 z_{uu}$$

$$\Longrightarrow z_{xx} = y^2 z_{uu}$$

Substituting these values in $\frac{\partial^2 z}{\partial x^2} + 2xy^2\frac{\partial z}{\partial x} + 2(y - y^3)\frac{\partial z}{\partial y} + x^2y^2z = 0$, we get

$$y^2 z_{uu} + 2y^2 uz_u + 2(1-y^2).uz_u - vz_v + x^2y^2z = 0$$

$$\Longrightarrow \frac{1}{v^2}z_{uu} + 2\frac{1}{v^2}uz_u + 2\left(1 - \frac{1}{v^2}\right).(uz_u - vz_v) + u^2z = 0$$

$$\Longrightarrow z_{uu} + 2uz_u + 2(v^2-1).(uz_u - vz_v) + u^2v^2z = 0$$

$$\Longrightarrow z_{uu} + 2uv^2z_u + 2(v - v^3).z_v + u^2v^2z = 0$$

Problem: If $U = F(x-y, y-z, z-x)$, then prove that $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$

Sol: $U = F(x-y, y-z, z-x)$

Suppose $x - y = r,\ y - z = s, z - x = t$, then $U = F(r,s,t)$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial U}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial U}{\partial r}.1 + \frac{\partial U}{\partial s}.0 + \frac{\partial U}{\partial t}.-1 = \frac{\partial U}{\partial r} - \frac{\partial U}{\partial t}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial U}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial U}{\partial r}.-1 + \frac{\partial U}{\partial s}.1 + \frac{\partial U}{\partial t}.0 = -\frac{\partial U}{\partial r} + \frac{\partial U}{\partial s}$$

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial U}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial U}{\partial r}.0 + \frac{\partial U}{\partial s}.-1 + \frac{\partial U}{\partial t}.1 = -\frac{\partial U}{\partial s} + \frac{\partial U}{\partial t}$$

Therefore $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = 0$.

Definition: If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Similarly the Jacobian of u, v, w with respect to x, y, z is $\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

Problem: If $x = u(1-v), \;\; y = uv$, then find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol: $x = u - uv = u - y \Rightarrow u = x + y$ and $v = \frac{y}{u} = \frac{y}{x+y}$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}, \frac{\partial v}{\partial y} = \frac{(x+y).1 - y.1}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{1}{x+y} = u$$

Problem: If $u = x + y + z, uv = y + z, \;\; uvw = z$, show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Sol : $u = x + y + z \quad \Rightarrow x = u - (y + z) = u - uv$

$uv = y + z \quad \Rightarrow y = uv - z = uv - uvw$ and $z = uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v - vw & u - uw & -uv \\ vw & uw & uv \end{vmatrix} R_1 \to R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v - vw & u - uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= uv(u - uw) - u^2 vw$$

$$= u^2 v$$

Problem: If $u = \tan^{-1} x + \tan^{-1} y, \;\; v = \frac{x+y}{1-xy}$, then find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol:

$$\frac{\partial u}{\partial x} = \frac{1}{1+x^2}, \frac{\partial u}{\partial y} = \frac{1}{1+y^2}, \frac{\partial v}{\partial x} = \frac{(1-xy).1-(x+y).-y}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}, \frac{\partial v}{\partial y} = \frac{(1-xy).1-(x+y).-x}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix} = \frac{1}{1+x^2} \cdot \frac{1+x^2}{(1-xy)^2} - \frac{1}{1+y^2} \cdot \frac{1+y^2}{(1-xy)^2} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Problem: If

$x_r \quad x_\theta \quad x_\varphi \qquad \sin\theta\cos \qquad r\cos\theta\cos \qquad -r\sin\theta\sin$

$r \quad \theta \quad \varphi \qquad \cos \qquad -r$

$x = r\sin\theta\cos\varphi\,, y = r\sin\theta\sin\varphi\,, z = r\cos\theta$, then find $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$

Sol:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} & & \\ y_r & y_\theta & y_\varphi \\ z & z & z \end{vmatrix} = \begin{vmatrix} \varphi & \varphi & \varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \theta & \sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Properties of Jacobians: **1. If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$, then $JJ' = 1$**

2. Chain rule for Jacobians: If u, v are functions of r, s ans r, s are functions of x, y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

3. Jacobian of Implicit functions: If u_1, u_2, u_3 instead of being given explicitly in terms of x_1, x_2, x_3 be connected with them equations such as

$\boldsymbol{f_1(u_1,u_2,u_3,x_1,x_2,x_3) = 0}$,
$\boldsymbol{f_2(u_1,u_2,u_3,x_1,x_2,x_3) = 0, f_3(u_1,u_2,u_3,x_1,x_2,x_3) = 0}$, **then**

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)} \Big/ \frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)}$$

Problem: If $x = r\cos\theta\,, y = r\sin\theta,$ evaluate $J = \frac{\partial(x,y)}{\partial(r,\theta)}$ and $J' = \frac{\partial(r,\theta)}{\partial(x,y)}$. Also show that $JJ' = 1$.

Sol: $x = r\cos\theta$ and $v = r\sin\theta$ then

$\frac{\partial x}{\partial r} = \cos\theta\,, \frac{\partial x}{\partial \theta} = -r\sin\theta\,, \frac{\partial y}{\partial r} = \sin\theta\,, \frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{(x^2 + y^2)}$ and

$\frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{(x^2+y^2)}}2x = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{1}{2\sqrt{(x^2+y^2)}}2y = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}.-\frac{y}{x^2} = -\frac{y}{(x^2+y^2)} = -\frac{y}{r^2}, \frac{\partial \theta}{\partial y} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}.\frac{1}{x} = \frac{x}{(x^2+y^2)} = \frac{x}{r^2}$$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{1}{r}$$

Therefore $JJ' = r\frac{1}{r} = 1$

Taylor's theorem for functions of two variables:

The Taylor's series expansion for a function $f(x, y)$ in powers of $x - a$ and $y - b$ is

$$f\,(x, y) = f(a, b) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)] +$$
$$\frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \ ...$$

Note: The Taylor's series expansion for a function $f(x, y)$ in powers of x and y is

$$f\,(x, y) = f(0, 0) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)] +$$
$$\frac{1}{2!}[x^2 f_{xx}(a, b) + 2xy f_{xy}(a, b) + y^2 f_{yy}(a, b)] + \ ...$$

Problem: Expand $e^x \sin y$ in powers of x and y up to second degree term.

Sol: Suppose $f(x, y) = e^x \sin y,\ f(0,0) = 0$

$$f_x(x, y) = e^x \sin y,\ f_x(0,0) = 0$$
$$f_y(x, y) = e^x \cos y,\ f_y(0,0) = 1$$
$$f_{xx}(x, y) = e^x \sin y,\ f_{xx}(0,0) = 0$$
$$f_{xy}(x, y) = e^x \cos y,\ f_{xy}(0,0) = 1$$
$$f_{yy}(x, y) = -e^x \sin y,\ f_{yy}(0,0) = 0$$

Taylor's series expansion for a function $f(x, y)$ in the neighbourhood of (a, b)

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ \ldots$$

$$\Rightarrow f(x,y) = y + xy + \cdots$$

Problem: Expand $\tan^{-1}\frac{y}{x}$ in powers of $(x-1)$ and $(y-1)$ up to second degree terms.

Sol: $f(x,y) = \tan^{-1}\frac{y}{x}$ $\qquad f(1,1) = \frac{\pi}{4}$

$$f_x = \frac{1}{\left(1+\frac{y^2}{x^2}\right)} . -\frac{y}{x^2} = -\frac{y}{(x^2+y^2)}$$

$$f_x(1,1) = -\frac{1}{2}$$

$$f_y = \frac{1}{\left(1+\frac{y^2}{x^2}\right)} . \frac{1}{x} = \frac{x}{(x^2+y^2)} \qquad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{2xy}{(x^2+y^2)^2} \qquad f_{xx}(1,1) = \frac{1}{2}$$

$$f_{xy} = -\frac{\left((x^2+y^2).1 - y.2y\right)}{(x^2+y^2)^2} = -\frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$f_{xy}(1,1) = 0$$

$$f_{yy} = -\frac{2xy}{(x^2+y^2)^2} \qquad f_{yy}(1,1) = -\frac{1}{2}$$

The Taylor's series expansion for a function $f(x,y)$ in powers $(x-a)$ and $(y-b)$ is

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ \ldots$$

$$\Rightarrow f(x,y) = f(1,1) + \left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] +$$

$$\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \ \ldots$$

$$\Rightarrow f(x,y) = \frac{\pi}{4} + \left[(x-1).-\frac{1}{2} + (y-1).\frac{1}{2}\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}.\frac{1}{2} + (y-1)^2.-\frac{1}{2}\right] + \ \ldots$$

$$\Rightarrow f(x,y) = \frac{\pi}{4} - \frac{1}{2}[(x-1) - (y-1)] + \frac{1}{4}[(x-1)^2 - (y-1)^2] + \ \ldots$$

Maxima and mimina of functions of two variables: A function $f(x,y)$ is said to have maximum or minimum at (a,b) according as $f(a+h,b+k) < f(a,b)$ or $f(a+h,b+k) > f(a,b)$ for all positive or negative small vales of h and k.

Note: $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

i.e. the function is stationary at the point (a, b).

Working rule to find the maximum and minimum values: Suppose $f(x, y)$ be the given function.

1. find $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ and equate to each zero. Solve these as simultaneous equations in x and y.

Let $(a, b), (c, d), \ldots$ be the stationary points.

2. find the values of $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at each stationary point.

3. (i) If $rt - s^2 > 0$ and $r < 0$ at (a, b), then f has maximum value at (a, b).

(ii) If $rt - s^2 > 0$ and $r > 0$ at (a, b), then f has minimum value at (a, b).

(iii) If $rt - s^2 < 0$ at (a, b), then $f(a, b)$ is not an extreme value. i.e. (a, b) is saddle point.

(iv) If $rt - s^2 = 0$ at (a, b), it needs further investigation.

Problem: Find the maximum and minimum values of the function
$x^2 + y^2 - 2x$

Sol: $f(x, y) = x^2 + y^2 - 2x$

$f_x = 2x - 2$ and $f_y = 2y$

Suppose $f_x = 0$ and $f_y = 0$ i.e $2x - 2 = 0$ *and* $2y = 0 \Rightarrow x = 1$ *and* $y = 0$

$\therefore (1, 0)$ the stationary point.

Now $r = f_{xx} = 2$, $s = f_{xy} = 0$ and $t = f_{yy} = 2$

$r \text{ at } (1, 0) = 2$, $s \text{ at } (1, 0) = 0$ and $t \text{ at } (1, 0) = 2$

Consider $rt - s^2 = 4 - 0 = 4 > 0$ and $r = 2 > 0$

Therefore f has minimum value at $(1, 0)$ and $f_{min} = 1 + 0 - 2 = -1$

Problem: Discuss the maximum and minimum values of u if
$u = ax^3y^2 - x^4y^2 - x^3y^3$

Sol: $u = ax^3y^2 - x^4y^2 - x^3y^3$

$u_x = 3ax^2y^2 - 4x^3y^2 - 3x^2y^3$ Suppose

$u_y = 2ax^3y - 2x^4y - 3x^3y^2$

$u_x = 0 \Rightarrow 3ax^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3a - 4x - 3y) = 0$

$u_y = 0 \Rightarrow 2ax^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2a - 2x - 3y) = 0$ The

$\Rightarrow x = 0, y = 0, 2x + 3y = 2a$ stationary points are $(0,0), \left(\frac{a}{2}, \frac{a}{3}\right)$

$r = u_{xx} = 6axy^2 - 12x^2y^2 - 6xy^3 \Rightarrow r\ at\ \left(\frac{a}{2}, \frac{a}{3}\right) = -\frac{a^4}{9}$

$s = u_{xy} = 6ax^2y - 8x^3y - 9x^2y^2 \Rightarrow s\ at\ \left(\frac{a}{2}, \frac{a}{3}\right) = -\frac{a^4}{12}$

$t = u_{yy} = 2ax^3 - 2x^4 - 6x^3y \Rightarrow t\ at\ \left(\frac{a}{2}, \frac{a}{3}\right) = -\frac{a^4}{8}$

And $rt - s^2 = \frac{a^8}{72} - \frac{a^8}{144} = \frac{a^8}{144} > 0$ and $r < 0$

$\Longrightarrow u$ has a maximum value at $\left(\frac{a}{2}, \frac{a}{3}\right)$ and $u_{max} = \frac{a^6}{432}$

Problem: Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$

Sol: $u = x^3y^2 - x^4y^2 - x^3y^3$

$u_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$u_y = 2x^3y - 2x^4y - 3x^3y^2$

Suppose

$u_x = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0$

$u_y = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0$ $\Rightarrow x = 0, y = 0,\ 4x + 3y = 3$

$\Rightarrow x = 0, y = 0, 2x + 3y = 2$ The stationary points are $(0,0)\ and\ \left(\frac{1}{2}, \frac{1}{3}\right)$.

$r = u_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 \Rightarrow r\ at\ \left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{9}$

$s = u_{xy} = 6x^2y - 8x^3y - 9x^2y^2 \Rightarrow s\ at\ \left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{12}$

$t = u_{yy} = 2x^3 - 2x^4 - 6x^3y \Rightarrow t\ at\ \left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{8}$

And $rt - s^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$ and $r = -\frac{1}{9} < 0$

$\Longrightarrow u$ has a maximum value at $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $u_{max} = \frac{1}{432}$

Lagrange's method of undetermined multipliers:

Sometimes it is requires to find the stationary vales of a function of several variables which are not all independent but connected by some given relations. Generally, we convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method proves very convenient.

Let $u = f(x, y, z)$ … (1) be a function of three variables x, y, z which are connected by the relation

$\varphi(x, y, z) = 0$ …. (2)

For u to have stationary vales, it is necessary that $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0$

Differentiating (1), we get $\frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy + \frac{\partial u}{\partial z}\, dz = du = 0$ …. (3)

Also differentiating (2), we get $\frac{\partial \varphi}{\partial x}\, dx + \frac{\partial \varphi}{\partial y}\, dy + \frac{\partial \varphi}{\partial z}\, dz = d\varphi = 0$ …. (4)

Multiply (4) by parameter λ and adding to (3), we get

$$\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial \varphi}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y} + \lambda \frac{\partial \varphi}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z} + \lambda \frac{\partial \varphi}{\partial z}\right) dz = 0$$

This equation will be satisfied if $\frac{\partial u}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0, \frac{\partial u}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0, \frac{\partial u}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0$

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working rule: 1. Write $F = u(x, y, z) + \lambda\varphi(x, y, z)$

2. Obtain the equations $\frac{\partial F}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$

3. These three equations together with $\varphi(x, y, z) = 0$

The values of x, y, z so obtained will give the stationary value of $f(x, y, z)$.

Problem: In the plane triangle ABC, find the maximum value of

$\cos A \cos B \cos C$

Sol: $U(A, B, C) = \cos A \cos B \cos C$ and $\phi(A, B, C) = A + B + C - \pi = 0$

Define $F = \cos A \cos B \cos C + \lambda(A + B + C - \pi)$,

$$F_A = \frac{\partial F}{\partial A} = -\sin A \cos B \cos C + \lambda$$

$$F_B = \frac{\partial F}{\partial B} = -\sin B \cos A \cos C + \lambda$$

$F_C = \frac{\partial F}{\partial C} = -\sin C \cos B \cos A + \lambda$

Suppose $F_A = 0, F_B = 0, F_C = 0$. i.e.

$-\sin A \cos B \cos C + \lambda = 0 \Rightarrow \lambda = \sin A \cos B \cos C$

$\sin B \cos A \cos C + \lambda = 0 \Rightarrow \lambda = \sin B \cos A \cos C -$

$\sin C \cos B \cos A + \lambda = 0 \Rightarrow \lambda = \sin C \cos B \cos A -$ $\quad \Rightarrow \tan A = \tan B = \tan C$

$\Rightarrow A = B = C = \frac{\pi}{3} (\because A + B + C = \pi)$

$\lambda = \sin A \cos B \cos C = \sin B \cos A \cos C = \sin C \Rightarrow B \cos A \cos$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ is the stationary point of

$U(A, B, C)$ and $U_{max} = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$

Problem: A rectangular open box of capacity 32 cubic units is to be prepared. Find the dimensions of the box, to minimize the cost of painting outside.

Sol: Let $x, y \text{ and } z$ units be the sides of the box and S be its surface.

Then $S = xy + 2yz + 2zx$ and $V = xyz = 32$

Define $F(x, y, z) = xy + 2yz + 2zx + \lambda(xyz - 32)$

Then $\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \Longrightarrow \lambda = \frac{-(y+2z)}{yz}$

$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \Longrightarrow \lambda = \frac{-(x+2z)}{zx}$

$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \Longrightarrow \lambda = \frac{-(2y+2x)}{xy}$

$\Longrightarrow \lambda = \frac{-(y+2z)}{yz} = \frac{-(x+2z)}{zx} = \frac{-(2y+2x)}{xy}$

Now

$\frac{-(y+2z)}{yz} = \frac{-(x+2z)}{zx} \Longrightarrow x(y + 2z) = y(x + 2z) \Longrightarrow x = y$

and

$\frac{-(x+2z)}{zx} = \frac{-(2y+2x)}{xy} \Longrightarrow y(x + 2z) = z(2y + 2x) \Longrightarrow y = 2z$

Therefore $x = y = 2z = k \ (say)$

Since $xyz = 32 \Longrightarrow k.k.\frac{k}{2} = 32 \Longrightarrow k = 4$

Hence Shas minimum value when $x = 4, y = 4, z = 2$.

UNIT V

MULTIPLE INTEGRALS

Problem: Evaluate $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}} \, dydx$

Sol: $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}} \, dydx = \int_0^2 \left\{ \frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right\}_0^{x^2} dx$

$$= \int_0^2 x(e^x - 1) \, dx$$

$$= \left\{ e^x(x-1) - \frac{x^2}{2} \right\}_0^2 = e^2 - 1$$

Problem: Evaluate $\int_{-1}^2 \int_{x^2}^{x+2} dy \, dx$.

Sol: $I = \int_{-1}^2 \int_{x^2}^{x+2} dy \, dx = \int_{-1}^2 (y)_{y=x^2}^{y=x+2} \, dx$

$$= \int_{-1}^2 (x + 2 - x^2) \, dx$$

$$= \left\{ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right\}_{x=-1}^{x=2} = \frac{9}{2}$$

Problem: Evaluate $I = \int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol: The region of the integration is bounded by $y = 0 \, , y = 1, x = 0 \; and \; x = 1$

$$I = \int_0^1 \int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \{\sin^{-1} y\}_{y=0}^{y=1} = \frac{\pi}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \{\sin^{-1} x\}_{x=0}^{x=1} = \frac{\pi}{2} \frac{\pi}{2} = \frac{\pi^2}{4}$$

Problem: Evaluate $\int_0^1 \int_0^{x^2} e^x \, dydx$

Sol: $\int_0^1 \int_0^{x^2} e^x \, dydx = \int_0^1 e^x \, (y)_{y=0}^{y=x^2} \, dx$

$$= \int_0^1 e^x \, x^2 dx$$

$$= \{e^x . x^2 - e^x . 2x + e^x . 2\}_{x=0}^{x=1}$$

$$= e - 2$$

Problem: Evaluate $\iint_R \; e^{2x-3y} dxdy$ over the triangle bounded by $x = 0\,, y = 0 \; and \; x + y = 1$.

Sol: Given region of integration is triangle formed by lines $x = 0, y = 0 \; and \; x + y = 1$.

The line $x + y = 1$ intersects x-axis at $(1,0)$ and y-axis at $(0,1)$.

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} e^{2x-3y} dxdy = \int_0^1 \left\{ \frac{e^{2x-3y}}{-3} \right\}_{y=0}^{y=1-x} dx$$

$$= -\frac{1}{3}\int_0^1 (e^{2x-3(1-x)} - e^{2x})dx$$

$$= -\frac{1}{3}\int_0^1 (e^{5x-3} - e^{2x})dx$$

1

$$= -\frac{1}{3}\left(\frac{e^{5x-3}}{5} - \frac{e^{2x}}{2}\right)_0$$

$$= -\frac{1}{3}\left[\left(\frac{e^2}{5} - \frac{e^2}{2}\right) - \left(\frac{1}{5} - \frac{1}{2}\right)\right] = \frac{e^2-1}{10}$$

Problem: Evaluate the $\iint y \, dxdy$ over the region R where R is the region bounded by parabolas $y^2 = 4x$ and $x^2 = 4y$.

Sol: Given R is the region bounded by parabolas $y^2 = 4x and \; x^2 = 4y$

The point of intersection of given two parabolas is $(4,4)$.

$$I = \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{2\sqrt{y}} y \; dx \; dy \; = \int_{y=0}^{4} y\{x\}_{x=\frac{y^2}{4}}^{x=2\sqrt{y}} \; dy$$

$$= \int_0^4 y\left(2\sqrt{y} - \frac{y^2}{4}\right) dy$$

=4

$$= \left\{2\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{16}\right\}_{y=}^{y}$$

0

$$= \frac{4}{5}.32 - 16 = \frac{48}{5}$$

Problem: Evaluate $\int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x+y} x \; dzdydx$.

Sol: $I = \int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x+y} x \; dzdydx = \int_{x=0}^{1} \int_{y=0}^{x} x(z)_{z=0}^{z=x+y} \; dydx$

$$= \int_{x=0}^{1} \int_{y=0}^{x} x(x + y) \; dydx$$

$$= \int_{x=0}^{1} \left(x^2 y + x\frac{y^2}{2}\right)_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{3}{2} x^3\, dx$$

$$= \frac{3}{2}\left(\frac{x^4}{4}\right)_{x=0}^{x=1} = \frac{3}{8}$$

Problem: Evaluate $\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{2} x^2\, yzdxdydz$

Sol: $I = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{2} x^2\, yzdxdydz = \int_{x=0}^{1} \int_{y=0}^{2} x^2\, y\left(\frac{z^2}{2}\right)_{z=0}^{z=2} dxdy$

$$= \int_{x=0}^{1} \int_{y=0}^{2} 2x^2\, ydxdy$$

$$= \int_{x=0}^{1} 2x^2 \left(\frac{y^2}{2}\right)_{y=0}^{y=2} dx$$

$$= \int_{x=0}^{1} 4x^2\, dx$$

$$= 4\left\{\frac{x^3}{3}\right\}_{x=0}^{x=1} = \frac{4}{3}$$

Problem: Evaluate $\int_{y=1}^{e} \int_{x=1}^{log\ y} \int_{z=1}^{e^x} \log z\ dz\ dy\ dx$.

Sol: $I = \int_{y=1}^{e} \int_{x=1}^{log\ y} \int_{z=1}^{e^x} \log z\ dz\ dy\ dx = \int_{y=1}^{e} \int_{x=1}^{log\ y} \{z(\log z - 1)\}_{z=1}^{z=e^x}\ dydx$

$$= \int_{y=1}^{e} \int_{x=1}^{log\ y} [e^x (x-1) + 1]\, dxdy$$

$$= \int_1^e \{e^x (x-1) - e^x + x\}_{x=1}^{x=\log y}\ dy$$

$$= \int_1^e (y(\log y - 1) - y + \log y) - (-e+1)\, dy$$

$$= \int_1^e (y \log y - 2y + \log y + e - 1) dy$$

$$= \left(\frac{y^2}{2}\log y - \frac{y^2}{4} - y^2 + y(\log y - 1) + (e-1)y\right)_1^e$$

$$= \left(\frac{e^2}{2} - \frac{e^2}{4} - e^2 + e(e-1)\right) - \left(0 - \frac{1}{4} - 1 + 0 + (e-1)\right)$$

$$= \frac{e^2+9}{4} - 2e$$

Problem: Evaluate $\int_{z=0}^{1}\int_{y=0}^{2}\int_{x=0}^{2} x^2\, yz\, dzdydx$

Sol: $I = \int_{z=0}^{1}\int_{y=0}^{2}\int_{x=0}^{2} x^2\, yzdzdydx \;\; = \int_{z=0}^{1}\int_{y=0}^{2}\left\{\frac{x^3}{3}\right\}_{x=0}^{x=2} yzdzdy$

$$= \frac{8}{3}\int_{z=0}^{1}\left\{\frac{y^2}{2}\right\}_{y=0}^{y=2} dz$$

$$= \frac{16}{3}\int_{z=0}^{1} zdz$$

$$= \frac{16}{3}\left\{\frac{z^2}{2}\right\}_{z=0}^{z=1} = \frac{8}{3}$$

Problem: Evaluate $\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$ by changing the order of integration.

Sol: The region of the integration is bounded by
$y = x^2/4a, y = 2\sqrt{ax}, x = 0 \;and\; x = 4a$

On changing the order of integration, $x \to \frac{y^2}{4a}\;\; to\; 2\sqrt{ay}$ and $y \to 0\; to\; 4a$

$I = \int_{0}^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dxdy = \int_{0}^{4a}\{x\}_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} dy$

$$= \int_{0}^{4a}\left(2\sqrt{a}y - \frac{y^2}{4a}\right) dy$$

$$4a \quad = \left\{2\sqrt{a}\frac{y^2}{2} - \frac{1}{4a}\frac{y^3}{3}\right\}_{y=0}^{y=}$$

Problem: Evaluate $\int_0^\infty\int_0^\infty \mathrm{e}^{-(x^2+y^2)}\, dxdy$ by changing to polar coordinates.

Sol: $I = \int_0^\infty\int_0^\infty \mathrm{e}^{-(x^2+y^2)}\, dxdy$

Area of the given integration is bounded by $y = 0, y = \infty, x = 0, x = \infty$

Put $x = r\cos\theta\; y = r\sin\theta$, then $dxdy = |J|drd\theta = rdrd\theta$

$I = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\, rdrd\theta = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-t}\frac{dt}{2}d\theta \;\; (\because r^2 = t \Rightarrow 2rdr = dt)$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\{-e^{-t}\}_{t=0}^{t=\infty}\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

Problem: Evaluate $\int_0^1\int_y^1 e^{-x^2}\, dxdy$

Sol: $I = \int_0^1 \int_y^1 e^{-x^2}\, dxdy$

On changing the order of integration $y \to 0 \ \ to\, x$ and $x \to 0 \ \ to\ 1$.

$$I = \int_0^1 \int_y^1 e^{-x^2}\, dxdy = \int_0^1 \int_0^x e^{-x^2}\, dxdy$$

$$= \int_0^1 e^{-x^2} \{y\}_{y=0}^{y=x} dx$$

$$= \int_0^1 e^{-x^2}\, xdx = \frac{1}{2}\left(1 - \frac{1}{e}\right)$$

Problem: Evaluate the double integral

$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{(\ ^4\ \ \ ^2\ ^2)^{\frac{1}{2}}}\, dxdy$ by changing the $y\ \ -a\ \ x$ order of integration.

Sol: The region of the integration is bounded by

$y = \sqrt{ax}\ i.e\ y^2 = ax, y = a, x = 0\ and\ x = a$.

On changing the order of integration, $x \to 0\ to \frac{y^2}{a}$ and $y \to 0\ to\ a$

$$I = \int_0^a \int_0^{\frac{y^2}{a}} \frac{y^2}{(y^4 - a^2x^2)^{\frac{1}{2}}}\, dx\, dy = \frac{1}{a}\int_0^a \int_0^{\frac{y^2}{a}} \frac{y^2}{\left(\frac{y^4}{a^2} - x^2\right)^{\frac{1}{2}}}\, dx\, dy = \frac{1}{a}\int_0^a y^2 \left\{\frac{1}{\frac{y^2}{a}} \sin^{-1}\left(\frac{x}{\frac{y^2}{a}}\right)\right\}_{x=0}^{x=\frac{y^2}{a}} dy$$

$$= \int_0^a y^2 \left(\sin^{-1} 1 - \sin^{-1} 0\right) dy$$

$$= \int_0^a \frac{\pi}{2} y^2 dy$$

$$= \frac{\pi}{2}\left(\frac{y^3}{3}\right)_{y=0}^{y=a} = \frac{\pi a^3}{6}$$